STABLE POPULATION THEORY WITH TIME VARYING IMMIGRATION
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STABLE POPULATION THEORY
WITH TIME VARYING IMMIGRATION

1. INTRODUCTION

The effect of immigration has become of some importance. Originally, stable population theory was developed to study so called closed populations (Lotka (1939), Sharpe and Lotka (1911)). Over the past quarter of a century or so open populations have been investigated and the effect of immigration on the long-run behaviour has been examined.

Keyfitz (1971) investigated the long-run effect of emigration on a population while Coale (1972) considered the reduction in fertility needed to counterbalance the effect of a steady stream of immigrants. Epenshade et al. (1982) and Mitra (1983) analysed the consequences of a constant indefinite stream of immigrants while Cerone (1986) extended stable population theory to include a constant stream of immigrants.

Mitra (1990) examined the vital rates and the age structure of a long-term stationary population brought about by the effect of a constant stream of immigrants on below replacement fertility regimen of the local population. Smertmann (1990) investigated the rejuvenating force reflecting through the eventual age - structure of the ensuing population under similar conditions as Mitra (1990). Blanchet (1989) examines the possibility of regulating the age - structure of an ensuing population under a constant stream of immigrants.

General immigration models have been developed in the past. Sivamurthy (1982) used a discrete formulation of a Leslie matrix and the population was projected. An integral equation model was developed by Langhaar (1972) which may be solved numerically. In this article however, some simple models shall be developed in order to investigate the effect of time variation of immigration levels. Further, a number of authors including Feichtinger and Steinmann (1992) and Friedlander and Feldmann (1993), have indicated that the adoption of local fertility behaviour by the immigrant population to be unrealistic. A model is developed and analysed that allows for a gradual transition of the fertility behaviour of an immigrant population to that of the local population.

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The modelling in this article is done through the use of the Sharpe-Lotka single-sex integral population model to determine the eventual population behaviour using the conventional method of Laplace transforms.

The outline of the paper is as follows:

Section 2 develops the general model that allows for time varying immigration and vital rates. Expressions for the total birth rate, $B(t)$, the age-distribution $A(x,t)$ and the total population numbers $N(t)$ are presented. Section 3 presents and analyses models which allow the immigration behaviour to vary with time. The asymptotic or long-term behaviour of the total births is obtained using some simple but instructive time varying immigrant regimen. A model that allows the immigrant maternity behaviour to gradually approach that of the local population is analysed in Section 4. A numerical example demonstrating the transient behaviour of the total births and its approach towards the predicted asymptotic is demonstrated in Section 5. In Section 6 the asymptotic or large time population numbers and age - distribution are detailed together with the limiting age-density for the models presented in Sections 3 and 4.

2.   MODEL DEVELOPMENT

A general integral equation model was developed by Langhaar (1972) which includes time varying migration and vital rates. He was able to solve equations numerically. We shall only consider the female portion of the population as is customary. If we let $m(x,t)$ be the expected number of female offspring, to a woman aged x at time t, over the interval $(x, x+ dx)$ then the number of female births $B(t)$ is given by (Langhaar (1972))

$$B(t) = \int_0^\infty A(x,t)\, m(x,t) dx \quad , t \geq 0 \tag{1}$$

where

$$A(x,t) = B(t-x)\, l(x,t) + J(x,t) , \tag{2}$$

is the age-distribution, $l(x,t)$ is the time dependent survivor function and

$$J(x,t) = \psi(x,t)\, l(x,t), \tag{3}$$

with $J(x,t)dx$ being the number of first generation immigrants at time t within the age group $(x, x + dx)$.

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It should be noted at this stage that the integral in equation [1] is integrated between zero and infinity for convenience; in effect $m(x,t)$ is identically zero outside of the reproduction interval (α, β).

It should further be highlighted that equation [2] holds for $t \geq x$ and there is a lack of information about the population prior to $t = 0$, our chosen origin. Following Cerone (1986) from equations [1] and [2] (and using [3]) gives

$$B(t) = G(t) + F(t) + \int_0^t B(t-x)\Phi(x,t)dx \qquad [4]$$

where

$$G(t) = \int_0^\infty J(x,t)\, m(x,t)dx = \int_0^\infty \psi(x,t)\Phi(x,t)dx \qquad [5]$$

and

$$F(t) = \int_t^\infty B(t-x)\Phi(x,t)dx = \int_0^\infty B(-x)\Phi(x+t,t)dx.$$

Hence from [2]

$$F(t) = \int_0^\infty \left[\frac{A(x,0) - J(x,0)}{l(x,0)}\right]\Phi(x+t,t)dx$$

or

$$F(t) = \int_t^\infty \left[\frac{A(x-t,0) - J(x-t,0)}{l(x-t,0)}\right]\Phi(x,t)dx. \qquad [6]$$

The total population numbers $N(t)$ may be obtained by simply summing the combined age-distribution over all ages to give from [2]

$$N(t) = \int_0^\infty B(x,t)\, l(x,t)dx + \int_0^\infty J(x,t)dx.$$

Further, the age-density may also be determined as the ratio of the age-distribution over the total population, viz.

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$$a(x,t) = \frac{A(x,t)}{N(t)}.$$

In the subsequent work we shall concentrate on the solution for the total births $B(t)$ since once this is known then the total population numbers and age-density may simply be obtained from the above expressions.

Further, in this work it will be assumed that the survivor function of the host population will immediately be adopted by the immigrant population.

3. TIME VARYING IMMIGRATION MODELS

The basic model [4] - [6] will now be modified to determine the long-term or asymptotic effects of: a terminating stream of immigrants, an indefinite stream of immigrants with fluctuations, a sequence of terminating streams of immigrants and, a gradual change in the immigrant population. We shall assume, in order to highlight the effects of migration streams, that the net maternity function of both the immigrant and host population are independent of time and are identical.

Thus the equations [4] - [6] become

$$B(t) = G(t) + F(t) + \int_0^t B(t - x)\phi(x)dx \qquad [7]$$

where

$$G(t) = \int_0^\infty \psi(x,t)\phi(x)dx \qquad [8]$$

The solution, and in particular the asymptotic behaviour, of the births represented by the models [7] and [8] may be approached in the usual way using Laplace transform techniques. Thus, formally taking the Laplace transform of [7] would give upon rearrangement that

$$B^*(p) = \frac{G^*(p) + F^*(p)}{1 - \phi^*(p)}, \qquad [9]$$

from which $B(t)$ may be determined (Lopez (1961), Pollard (1973)) by considering the contribution, through the theory of residues in complex

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variables, from the poles of [9]. In a closed population (one without migration) Lopez (1961) showed that the poles were effectively the roots of

$$\phi^*(p) = 1 . \tag{10}$$

This is no longer necessarily the case here since there may be a contribution from the $G^*(p)$ term in [9]. This point was also made in Cerone (1986) where there was a contribution from a pole at $p = 0$ resulting from the model of a continuing constant stream of immigrants.

A number of immigration models will now be presented and their effect on the asymptotic behaviour of the births will be determined. It must be highlighted at this stage that the models presented are simple in nature to best be able to show the effect on the ensuing population.

3.1 A terminating stream of immigrants

We shall determine the consequences of a constant stream of immigrants (treated previously by Cerone (1986), Espenshade *et al.* (1982), Mitra (1983)) which will now be considered as coming to an indefinite halt after τ years. Thus, the model to be examined may be presented by equations [7] and [8] with

$$\psi(x,t) = S(x)H(\tau - t)H(t), \quad \tau > 0 \tag{11}$$

where $H(u)$ is the Heaviside unit function defined as unity for $u > 0$ and zero for $u < 0$.

That is, we need to solve [7] with $G(t)$ from [8] and [11] given by

$$G(t) = b_I H(\tau - t)H(t) \tag{12}$$

where

$$b_I = \int_0^\infty S(x)\phi(x)dx.$$

The long-term asymptotic behaviour will now be determined using traditional Laplace transform techniques.

Formally, following the Laplace transform approach we obtain, from [7] or else directly from [9] and using [12]

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$$B^{*}(p) = b_{I} \cdot \frac{1 - e^{-\tau p}}{p(1 - \phi^{*}(p))} + \frac{F^{*}(p)}{1 - \phi^{*}(p)}. \qquad [13]$$

It should be noted that for $\phi(x)$ a positive function, it can be shown (Pollard, 1973) that there is one real root r, of [10] greater that the real part of the complex ones, and so we have (Bellman and Cooke, 1963, Lopez, 1961) the asymptotic or large-time behaviour given by

$$B(t) \sim \lim_{p \to r} e^{pt}(p - r)B^{*}(p).$$

Hence, we obtain by procuring the contribution from the real pole, at $p=r$, the real root of [10],

$$B(t) \sim \begin{bmatrix} \left[b_{I}\left(1 - e^{-\tau r}\right) + rF^{*}(r)\right]\dfrac{e^{rt}}{r\kappa} &, & r \neq 0 \\[3mm] \dfrac{\tau b_{I} + F^{*}(0)}{\mu} &, & r = 0 \end{bmatrix} \qquad [14]$$

where,

$\kappa = \int_{0}^{\infty} e^{-rx} x \phi(x)dx$ is the expected age of giving birth in the eventual stable population

and $\mu = \int_{0}^{\infty} x \phi(x)dx$ is the expected age of giving birth in the eventual stationary population.

It should be noted that $p = 0$, although a zero of the denominator of the first term in [13], is not a pole since the numerator is also zero (we obtain a zero contribution from $p = 0$).

The result given by [14] is similar in *form* to a model without any immigration whatsoever ($b_{I} = 0$ or $\tau = 0$). This should not be too surprising since it is equivalent to a shift of the origin to $t = \tau$ at which immigration will have ceased. Thus, if $r < 0$ then an eventual extinction of the population would still result. Further, if we allow τ to tend to infinity then the results of an indefinite constant stream of immigrants are obtained in agreement with Cerone (1986).

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It is worthwhile noting that, from [14], we may answer the questions as to how long a constant stream of immigration would have to be continued in order to approach a certain stationary population. Thus, given a desired eventual number of births $B(t) \sim B_\infty$ we may determine from [14] that the constant stream b_I of immigrants should be terminated when

$$\tau \, b_I = \mu B_\infty - F^*(0).$$

The problem could also be solved if the desired eventual numbers in a population were specified.

This type of "Converse" problem was also examined by Cerone (1986) when dealing with changes in the maternity behaviour of a population.

3.2 An indefinite stream of immigrants with fluctuations

We now look at the effects of an indefinite constant underlying stream of immigrants with fluctuations at certain times. The model may be represented by [7], [8] with

$$\psi(x,t) = S_0(x) + \sum_{n=1}^{N} S_n(x) \delta(t - \tau_n) \tag{15}$$

where $\delta(u)$ is the Dirac delta function defined as non-zero only at $u = 0$.
That is,

$$G(t) = \beta_0 + \sum_{n=1}^{N} \beta_n \delta(t - \tau_n) \tag{16}$$

where

$$\beta_n = \int_0^\infty S_n(x) \phi(x) dx, \; n = 0, 1, \ldots \tag{17}$$

We note that the model represented by [15] may allow changes, in immigration at certain times τ_n, which may correspond to some government decision. For example, if $S_0(x)$ is the underlying stream of immigrants and some decision is made that at $t = \tau_I$, a change of $S_I(x)$ is to be made then, from [15],

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$$\psi(x,t) = \begin{cases} S_0(x) & , \quad t \neq \tau_1 \\ S_0(x) + S_1(x) & , \quad t = \tau_1 \,. \end{cases}$$

Thus, the model represented by [15] allows "once-off" changes at $t = \tau_1,....., \tau_N$ to an underlying immigration regime represented by $S_0(x)$.

We now follow a similar procedure as for the previous example to obtain the consequences of the above immigration policy as exemplified by equation [16].

Taking Laplace transforms of [7], or directly from [9], and using [16] gives, upon rearrangement,

$$B^*(p) = \frac{\beta_0}{p(1 - \phi^*(p))} + \frac{F^*(p) + \sum_{n=1}^{N} \beta_n e^{-\tau_n p}}{1 - \phi^*(p)} \,. \qquad [18]$$

We note that [18] is similar in form to equation [6] of Cerone (1986) so that the asymptotic behaviour of $B(t)$ is readily seen to be given by

$$B(t) \sim \begin{cases} \dfrac{\beta_0}{1 - R}, & r < 0 \\[3ex] \dfrac{\beta_0}{\mu_1}\left[t + \dfrac{1}{2}\mu_2\right] + \dfrac{1}{\mu_1}\left(F^*(0) + \sum_{n=1}^{N}\beta_n\right), & r = 0 \\[3ex] \dfrac{\beta_0}{1 - R}\left[\beta_0 + r\left(F^*(r) + \sum_{n=1}^{N}\beta_n e^{-\tau_n r}\right)\right]\dfrac{e^{rt}}{rk}, & r > 0 \end{cases} \qquad [19]$$

where $\mu_i = \dfrac{M_i}{M_{i-1}}, \qquad i = 1,2$

with $M_i = \int_0^\infty x^i \phi(x)dx$, the ith moment of ϕ and $R = M_0$ the net production rate.

A number of interesting points may be noted from the above model. If $\beta_n = 0$ (that is $S_n(x) \equiv 0$) for $n = 1, 2, \ldots, N$, then we obtain the indefinite

constant stream of immigration model obtained by Cerone (1986). If further $\beta_0 = 0$, we get the normal closed population model without any immigration.

Also, if $\beta_0 = 0$ and $\beta_n \neq 0$ for $n = 1, 2, \ldots, N$, then there is no underlying constant stream of immigration but, there are $S_n(x)\ l(x)$ immigrants of age x, at time $t = \tau_n$ for $n = 1, 2, \ldots, N$. The population would then tend towards a stationary state if replacement rates existed, that is, if $r = 0$.

We may again address ourselves to the converse problem of; given a desired eventual level of total births in the stationary population $B(t) \sim B$ what will the immigration parameters need to be? From [19] we see that

$$\beta_0 = B_\infty (1 - R)\ , \qquad r < 0, \beta_0 \neq 0 \tag{20a}$$

and $\quad \displaystyle\sum_{n=1}^{N} \beta_n = \mu_1 B_\infty - F^*(0), \qquad r = 0, \quad \beta_0 = 0. \tag{20b}$

From equation [20a] we see that the stationary population is obtained if $r < 0$ and $\beta_0 = 0$ (that is $S_0(x) \equiv 0$). It is the contribution to the births from the spasmodic immigrants and not the timing of their arrival which determines the level of the eventual stationary population. This, at first surprising result, may be explained by the fact that no matter when these migrants arrive, they will still have made the same contribution towards the eventual stationary population.

3.3 A sequence of terminating streams of immigrants

Building on to the results of the previous model it would be worthwhile investigating the effects of changes in the level of immigration as a change in policy is implemented. This is a realistic model as happens when for example there is a change of government and immigration policy is altered.

The model to be considered may be represented by [7], [8] with

$$\psi(x,t) = \sum_{n=0}^{N-1} S_n(x) H(\tau_{n+1} - t) H(t - \tau_n)$$

where H is the Heaviside unit function defined earlier and so

$$G(t) = \sum_{n=0}^{N-1} \beta_n(x) H(\tau_{n+1} - t) H(t - \tau_n) \tag{21}$$

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with β_n given by [17].

The Laplace transform of [21] is given by

$$G^*(p) = \sum_{n=0}^{N-1} \beta_n \frac{e^{-\tau_n p} - e^{-\tau_{n+1} p}}{p}. \qquad [22]$$

For τ_i finite and not all zero, we notice that $p = 0$ is a removable singularity of $G^*(p)$ so that the only contribution to the asymptotic behaviour comes from the real root r of [10] thus giving, from [9], that

$$B(t) \sim \begin{cases} \left(G^*(r) + F^*(r)\right)\dfrac{e^{rt}}{r} & , \quad r \neq 0 \\[2em] \dfrac{G^*(0) + F^*(0)}{\mu} & , \quad r = 0. \end{cases}$$

That is, upon using [22] the asymptotic behaviour of the birth rate is

$$B(t) \sim \begin{cases} \left(\displaystyle\sum_{n=0}^{N} \gamma_n e^{-\tau_n r} + r F^*(r)\right)\dfrac{e^{rt}}{r\kappa} & , \quad r \neq 0 \\[2em] \left(-\displaystyle\sum_{n=0}^{N} \gamma_n \tau_n + F^*(0)\right)\Big/ \mu & , \quad r = 0 \end{cases} \qquad [23]$$

where

$$\begin{aligned} \gamma_0 &= \beta_0 \\ \gamma_j &= \beta_j - \beta_{j-1}, \quad j = 1, 2, \ldots, N-1 \\ \gamma_N &= -\beta_N. \end{aligned}$$

If τ_N were allowed to become infinite then the last sequence of immigrants is continued indefinitely and so a double pole would exist for the coefficient of β_{N-1}.

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3.4 *A gradual change in immigration*

In keeping with the desire to use simple models to extract insights into the effects of immigration on a population the following model is proposed. Instead of assuming that a constant age-distribution of immigrants persisted indefinitely we consider the situation in which the current age - distribution of immigrants $S_c(x)$ changes over time towards an eventual distribution $S_e(x)$. This may be modelled by

$$\psi(x,t) = S_e(x) + e^{-\nu t}(S_c(x) - S_e(x))$$

and so from [8]

$$G(t) = b_e + e^{-\nu t}(b_c - b_e), \quad \nu > 0, \tag{24}$$

where $b_c = \int_0^\infty S_c(x)\phi(x)dx$ and $b_e = \int_0^\infty S_e(x)\phi(x)dx$.

Equation [7] with [24] may be treated in the usual way through Laplace transforms and so from [9]

$$\cdot B^*(p) = \left[\frac{b_e}{p} + \frac{b_c - b_e}{p + \nu} + F^*(p)\right] \Big/ \left[1 - \phi^*(p)\right]. \tag{25}$$

Now obtaining the contribution to the births through the real poles (namely, $p = 0$, $p = -\nu$ and $p = r$) of $B^*(p)$ using residues gives

$$\frac{b_e}{1 - \phi^*(0)} + \frac{(b_c - b_e)e^{-\nu\tau}}{1 - \phi^*(-\nu)} + \left(\frac{b_e}{r} + \frac{b_c - b_e}{r + \nu} + F^*(r)\right)\frac{e^{rt}}{\kappa}.$$

If $r = 0$ then $p = 0$ is a double pole of the b_e coefficient in [25] then, following Cerone (1986) or Feller (1941), we obtain the contribution from the real poles given by

$$\frac{b_e}{M_1(0)}\left[t + \frac{1}{2}M_2(0)\right] + \frac{F^*(0)}{M_1(0)} + (b_c - b_e)\left[\frac{e^{-\nu t}}{1 - \phi^*(-\nu)} + \frac{1}{\nu M_1(0)}\right]$$

where $\quad M_n(\theta) = \dfrac{\int_0^\infty e^{\theta x} x^n \phi(x) dx}{\int_0^\infty e^{\theta x} x^{n-1} \phi(x) dx}.$

If $r = -\nu$, so that local population is stable and decreasing, then p = $-\nu$ would be a double pole of the $(b_c - b_e)$ coefficient in [25] which may be treated in a similar way to give the total contribution from the real poles as

$$\frac{b_e}{1 - \phi^*(0)} + \left[\frac{b_e}{-\nu} + (b_c - b_e)\left(t + \frac{1}{2} M_2(-\nu) \right) + F^*(-\nu) \right] \frac{e^{-\nu t}}{M_1(-\nu)}.$$

4. TIME VARYING IMMIGRANT MATERNITY BEHAVIOUR

All of the models treated thus far, such as Mitra (1990), in the literature have assumed that the immigrant population immediately assumes the host country's vital rates. It had been pointed out in the past by Feichtinger and Steinmann (1992), at least, that for immigrants from high birth rate countries the expectation of immediate adoption of the host country's low birth rates is unrealistic.

We now consider a model in which the maternity function of the immigrant population gradually approaches that of the local population. Before pursuing this however it may be noticed from equation [1] that the maternity behaviour of the local and immigrant population may be varied so that

$$B(t) = G_m(t) + F(t) + \int_0^t B(t-x)\Phi(x,t)dx$$

where

$$G_m(t) = \int_0^\infty J_m(x,t)m_m(x,t)dx = \int_0^\infty \psi(x,t)\Phi_m(x,t)dx,$$

$$J_m(x,t) = \psi(x,t)l_m(x,t),$$

$$F(t) = \int_t^\infty \left[\frac{A(x-t,0) - J_m(x-t,0)}{l(x-t,0)} \right] \Phi(x,t)dx$$

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and $\quad A(x,t) = B(t-x)l(x,t) + J_m(x,t).$

While the model represented by [26] allows for a general time-dependent immigration regime with differing time varying maternity behaviour between the local and immigrant populations, a number of simplifying assumptions will be made to highlight the effects of the model. We wish to examine the effect of differing maternity behaviour between the local and immigrant populations. We shall assume a constant stream of immigrants and the maternity behaviour of the local population to be independent of time. Thus, the model to be examined becomes

$$B(t) = G_m(t) + F(t) + \int_0^t B(t-x)\phi(x)dx \qquad [27]$$

where

$$G_m(t) = \int_0^\infty S(x)\Phi_m(x,t)dx. \qquad [28]$$

Now consider the situation represented by the model

$$\Phi_m(x,t) = \phi(x) + e^{-\lambda t}[\phi_m(x) - \phi(x)] \ , \quad \lambda > 0 \qquad [29]$$

in which the immigrant population approaches the maternity behaviour of the local population gradually over time. Substitution of [29] into [28] would give

$$G_m(t) = (1 - e^{-\lambda t})b + e^{-\lambda t}b_m \ , \qquad \lambda \geq 0 \qquad [30]$$

where $\quad b = \int_0^\infty S(x)\phi(x)dx$

and $\quad b_m = \int_0^\infty S(x)\phi_m(x)dx.$

Thus the model to be solved is given by [27] and [30].
This may be done using the usual technique of Laplace transforms to give

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$$B^*(p) = \left[\frac{b}{p} + \frac{(b_m - b)}{(p + \lambda)} + F^*(p) \right] \bigg/ \left[1 - \phi^*(p) \right].$$ [31]

It may be noticed that equation [31] is similar to [25] with the exception of the b constants. In [24] - [25] there is a variation in the age-distribution of immigrants and in [30] - [31] it is the maternity behaviour that varies.

Thus, the large-time asymptotic behaviour of $B(t)$ is given by, upon recalling that $\lambda > 0$ ($\nu > 0$),

$$B(t) \sim \begin{cases} \dfrac{b}{1 - R} & , \quad r < 0 \\[2ex] \dfrac{b}{M_1(0)} \left[t + \dfrac{1}{2} M_2(0) \right] + \left[\dfrac{(b_m - b)}{\lambda} + F^*(0) \right] \bigg/ M_1(0) & , \quad r = 0 \\[2ex] \dfrac{b}{1 - R} + \left[\dfrac{b}{r} + \dfrac{b_m - b}{r + \lambda} + F^*(r) \right] \dfrac{e^{rt}}{\kappa} & , \quad r > 0. \end{cases}$$ [32]

It should be noted that if λ is allowed to become infinite then the model of a constant indefinite stream of immigrants immediately, upon arrival, adopting the local fertility behaviour is obtained and the result as given by equation [12] in Cerone (1986) is obtained. The results in Cerone (1986) could also be reproduced by allowing b_m to tend to b. If on the other hand $b \to b_m$ then the immigrants never adopt the local maternity behaviour but continue on with their original country's behaviour. The attraction of a model such as [29] is both its simplicity and versatility in that different values of λ could be used to model varying behaviour of for example, different ethnic groups. The value of λ controls the rate at which the transition occurs.

Equation [29] allows for a uniform steady transition of the immigrant population's maternity behaviour towards that of the host population. This may not be the situation and so the models developed by Cerone (1996) for a closed population may be utilised here and thus we may take

$$\Phi_m(x,t) = \phi(x) + \xi(t) \left[\phi_m(x) - \phi(x) \right]$$

where $\xi(0) = 1$ and $\lim_{t \to \infty} \xi(t) = 0$. For example, Cerone (1996) expressed $\xi(t)$ as a sum of exponentials and demonstrated its versatility in representing a gradual transition between two maternity functions. It could for example, if

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conditions were favourable in their new country, represent an initial increase in their maternity behaviour before a gradual uniform transition to the local population maternity behaviour. An initial lowering of the maternity behaviour could also be handled by a sum of exponential model for $\xi(t)$.

5.   NUMERICAL EXAMPLE

The Volterra integral equations can in a fairly straight forward fashion be evaluated numerically however, it is argued, this sheds less light on the determinants of certain behaviour than simple models coupled with some analysis. Nonetheless [27] together with [30] was solved numerically (following the procedures of Linz (1969), DeBoor (1971) and, Campbell and Day (1971)). The maternity functions were specified using a triangular shape with maximum heights ζ. The immigrant age-distribution was derived from the total age-specific permanent settlers in Australia during 1992 (Shu and Khoo, 1992, p.41) with a sex ratio of males to females of 1.078. To simplify the evaluation of $F(t)$ the population is assumed to be initially stable.

A value of $\zeta = 0.053$, $\alpha = 12.5$, $\beta = 47.5$ and a constant force of mortality $\mu = 0.0014$ (OAGA (1995)) gives $R = 0.992$ and $r = -0.0002942$.

Figure 1 shows the transient behaviour of the births for $\lambda = 0$, $-5r$ and $-10r$ where the height of the immigrant maternity function is $\zeta_m = 0.2$ and is non-zero over the same interval as the locals. Further, the immigrant population is assumed to constitute 1% of the original total births. In this illustrative example it may be seen that because of the high percentage of immigrants the immigrant maternity behaviour does considerably affect the total births. A gradual transition to a decreasing local fertility regimen shows that eventually the population does return to a decreasing one after a transient halting of the decline. The unbroken curve does show that a reversal and maintenance of a change in direction may be accomplished through immigration depending on both the level and the vital rates of the immigrants.

The $r = 0$ and $r > 0$ situations will not be demonstrated here since the effect is not as pronounced as the $r < 0$ case. The births simply commence an increasing trend with $r = 0$ and the immigrants reinforce the increasing trend of the $r > 0$ situation.

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FIGURE 1

Scaled birth rate of the total population when r < 0.
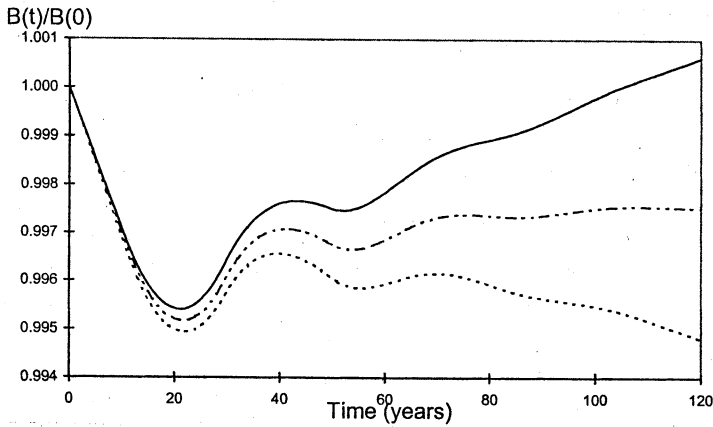
6. LONG TERM POPULATION AND AGE-DENSITY

The age-distribution of the time varying immigrant behaviour model is given by

$$A(x,t) = B(t-x)\, l(x) + J_m(x) \tag{33}$$

where $J_m(x) = S(x)\, l(x)$ since the survivor function is assumed identical to the local and the immigrant intake is constant.

The asymptotic behaviour is obtained from [32] to give [34]:

$$A(x,t) \sim \begin{cases} \left[\dfrac{b}{1-R} + S(x)\right] l(x) & ,\quad r < 0 \\[3ex] \left\{\dfrac{b}{M_1(0)}\left[t - x + \dfrac{1}{2}M_2(0)\right] + \left[\dfrac{b_m - b}{\lambda} + F^*(0)\right]\Big/ M_1(0) + S(x)\right\} l(x) & ,\quad r = 0 \\[3ex] \left\{\dfrac{b}{1-R} + \left[\dfrac{b}{r} + \dfrac{b_m - b}{r+\lambda} + F^*(r)\right]\dfrac{e^{rt}}{\kappa} + S(x)\right\} l(x) & ,\quad r > 0. \end{cases}$$

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Schmertmann (1992) investigated the effects of decreasing populations achieving stationarity through the action of a constant stream of immigrants.

The total population $N(t)$ is obtained from summing the age-distribution over all ages to give from [33],

$$N(t) = \int_0^\infty A(x,t)dx = \int_0^\infty [B(t-x) + S(x)]l(x)dx$$

and so the large - time asymptotic behaviour is obtained upon using [34] (or [32]) as [35]:

$$N(t) \sim \begin{cases} \dfrac{be_0^0}{1-R} + N_I & , \quad r < 0 \\[3ex] \dfrac{e_0^0}{M_1(0)}\left\{b\left[t + \dfrac{1}{2}M_2(0)\right] + \dfrac{b_m - b}{\lambda} + F^*(0)\right\} - \dfrac{b}{M_1(0)}\int_0^\infty xl(x)dx + N_I & , \quad r = 0 \\[3ex] \dfrac{be_0^0}{1-R} + \left(\dfrac{b}{r} + \dfrac{b_m - b}{r+\lambda} + F^*(r)\right)\dfrac{e^{rt}}{\widetilde{b}\,\kappa} + N_I & , \quad r > 0 \end{cases}$$

where $e_0^0 = \int_0^\infty l(x)dx$ is the expectation of life at birth,

$$\widetilde{b} = \left(\int_0^\infty e^{-rx}l(x)dx\right)^{-1}$$ is the intrinsic birth rate of a stable population

and $N_I = \int_0^\infty S(x)l(x)dx.$

If we allow λ to tend to infinity, then the immigrants immediately adopt the local fertility regimen and the results of Cerone (1986) are obtained. If λ tends to zero in [30] (equivalent to setting $b = b_m$ in [35]) then the overseas born women never adopt the local maternity behaviour.

The asymptotic or large - time behaviour of the age-density $a(x,t)$ may be obtained as equation [34] over equation [35].

The limiting age - density function $a(x)$ however, is given by

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$$a(x) = \lim_{t \to \infty} a(x,t) = \begin{cases} \left(\dfrac{b + (1-R)S(x)}{be_0^0 + (1-R)N_1} \right) l(x) & , \quad r < 0 \\[2em] \dfrac{l(x)}{e_0^0} & , \quad r = 0 \\[2em] \tilde{b} e^{-rx} l(x) & , \quad r > 0 \ , \end{cases} \qquad [36]$$

for $\lambda > 0$, which is identical to the density obtained in Cerone (1986). Thus, as long as the local maternity behaviour is approached then the long-term density is the same as that obtained by Cerone (1986). This at first surprising result may be reconciled by realising that the rate at which the density stabilizes would be different between the model of a general approach to the local maternity behaviour when compared to an abrupt adoption.

For $\lambda = 0$, on the other hand, the immigrant population never adopts the local maternity behaviour and the asymptotic births, age - distribution and numbers are obtained by putting $b = b_m$ in [32], [34] and [35] respectively so that the limiting age-density, is the same as [36] with b replaced by b_m.

The long-term or asymptotic effect on the total population numbers and age-distribution may be similarly obtained for the time - varying immigration models. It is interesting to note that for the models with only transient immigration behaviour (models (3.1) and (3.3)) then the limiting age density is

$$a(x) = \begin{cases} \dfrac{l(x)}{e_0^0} & , \quad r = 0 \\[2em] \tilde{b} e^{-rx} l(x) & , \quad r \neq 0. \end{cases}$$

The limiting behaviour is independent of the transient immigration regime even when $r < 0$. The limiting age-density for $r < 0$ for models (3.2) and (3.4) are similar in form to that obtained by Cerone (1986) and result in a limiting age-density that directly involves the indefinite or eventual stream of immigrants. For model (3.4), the expression for $r < 0$ as given in [36] is obtained with b replaced by B_e and $S_e(x)$ with $N_I = \int_0^\infty S_e(x)l(x)$. For model (3.2), β_0 replaces b is [36] and $S_0(x)$ replaces $S(x)$.

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7. CONCLUSIONS

Stable population theory has been extended to allow models of time-varying immigrant behaviour. Traditional methods of Laplace transforms have been applied to solve the Volterra single-sex integral population model for the total birth rate. Although the main emphasis was in developing expressions for the asymptotic effects of some simple but versatile models of time varying immigrant behaviour the transient, although to a certain extent less informative, solution was also demonstrated through an illustrative example.

A model has also been introduced that allows for the immigrants to gradually adopt the local maternity behaviour. A need for this has been expressed in the literature however it has not, to the best knowledge of the authors, hitherto been addressed.

The current article presents what is argued very versatile models for analysing the behaviour of time varying immigrant populations. It is left to policy and decision makers to decide on a prior regimen of immigrant intake.

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REFERENCES

BELLMAN R., COOKE K.L. (1963), *Differential Difference Equations*, Academic Press, New York.

BLANCHET D. (1989), "Regulating the Age Structure of a Population through Migration", *Population. English Selection*, (1), 23-37.

CAMPBELL G.M., DAY J.T. (1971), "A block by block method for the numerical solution of volterra integral equations", *BIT*, 11, 120-124.

CERONE P. (1986), "On Stable Population Theory with Immigration", *Demography*, 24, 431-438.

CERONE P. (1996), "On the Effects of the Generalised Renewal Integral Equation Model of Population Dynamics", *Genus,* Vol. LII n. 1-2, 53-70.

COALE A.J. (1972), *The Growth and Structure of Human Populations: A Mathematical Investigation*, New Jersey: Princeton University Press.

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DE BOOR C. (1971), "CADRE: An Algorithm for Numerical Quadrature", in RICE J.R. (ed.), Mathematical Software. New York: Academic Press, 417-443.

ESPENSHADE T.J., BOUVIER L.F., ARTHUR W.B., (1982), "Immigration and the Stable Population Model", *Demography*, 19, 125-133.

FEICHTINGER G., STEINMANN G. (1992), "Immigration into a Population with Fertility below Replacement Level - The Case of Germany", *Population Studies*, 46, 275-284.

FRIEDLANDER D., FELDMANN C. (1993), "The Modern Shift to Below-replacement fertility: Has Israel's Population Joined the Process?", *Population Studies*, 47, 295-306.

KEYFITZ N. (1971), "Migration as a Means of Population Control", *Population Studies*, 25, 63-72.

LANGHAAR H.L. (1972), "General Population Theory in the Age-Time Continuum", *Journal of the Franklin Institute*, 293, 199-213.

LINZ P. (1969), "A Method for Solving Nonlinear Volterra Integral Equations of the Second Kind", *Mathematical Computation*, 23, 595-599.

LOPEZ A. (1961), *Problems in Stable Population Theory*, Report, Princeton University, Office of Population Research.

LOTKA A.J. (1939), "A Contribution to the Theory of Self-Renewing Aggregates, with Special Reference to Industrial Replacement", *Annals of Mathematical Statistics*, 10, 1-25.

MITRA S. (1983), "Generalisation of the Immigration and the Stable Population Model", *Demography*, 20, 111-115.

MITRA S. (1990). "Immigration, Below-Replacement Fertility, and Long-Term National Population Trends", *Demography*, 27, 121-129.

OAGA - OFFICE OF THE AUSTRALIAN GOVERNMENT ACTUARY (1995), *Australian Life Tables 1990-92*. Canberra: Australian Government Publishing Service.

POLLARD J.H. (1973), *Mathematical Models for the Growth of Human Populations*. Cambridge: Cambridge University Press.

SCHMERTMANN C.P. (1992), "Immigrants' Age and the Structure of Stationary Populations with Below-Replacement Fertility", *Demography*, 29, 595-612.

SHARPE F.R., LOTKA A.J. (1911), "A Problem in Age-Distribution", *Philosophical Magazine*, 21, 435-438.

SHU J., KHOO S.E. (1993), *Australia's Population Trends and Prospects 1992*, Canberra: Australian Government Publishing Service.

SIVAMURTHY M. (1982), *Growth and Structure of Human Population in the Presence of Migration*, London, Academic Press.

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